Cryptanalyses de logarithmes discrets

Journées de la sécurité GDR 2022

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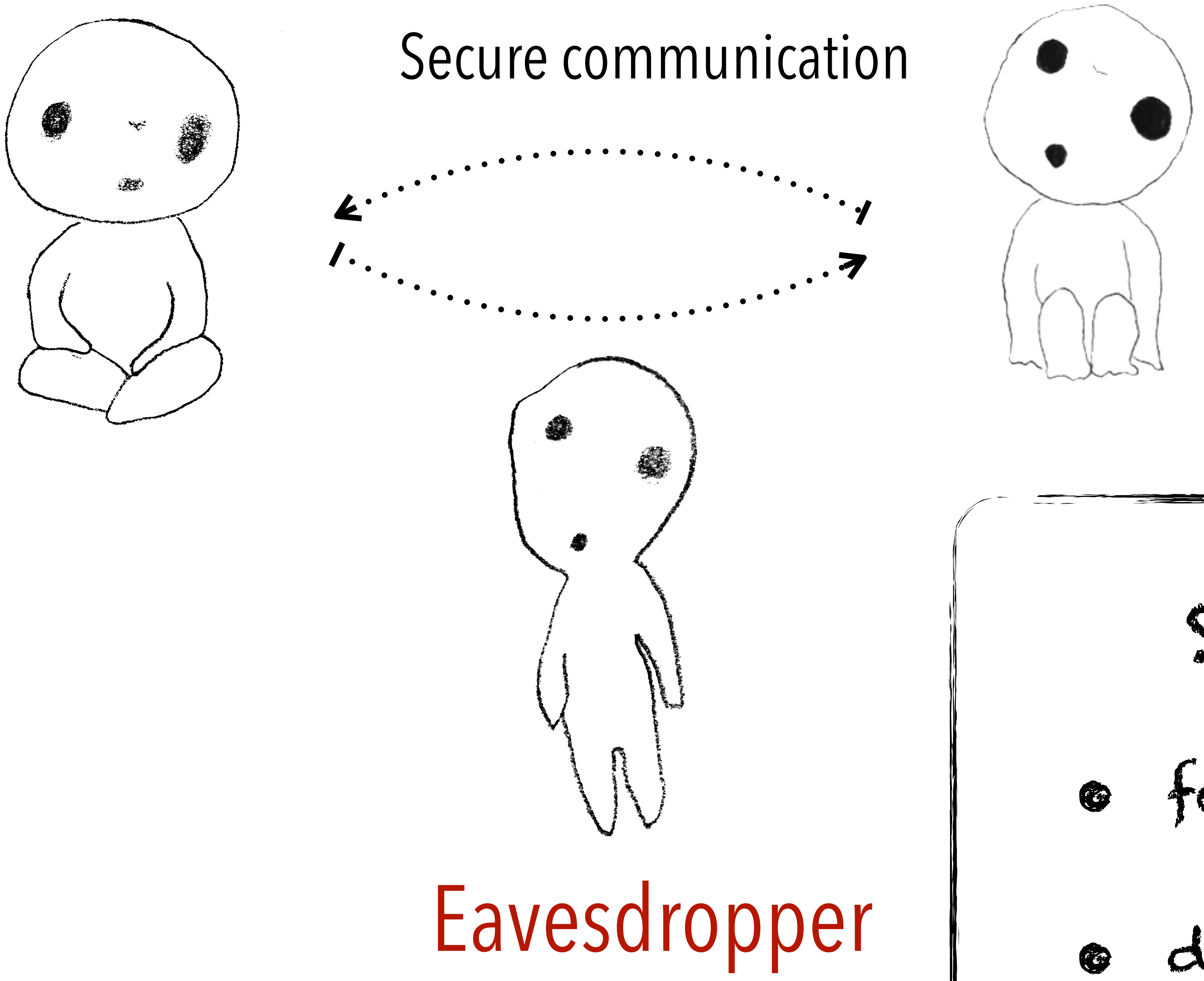
Hard problems for Cryptography



Use (hopefully) intractable problems to construct cryptographic primitives.

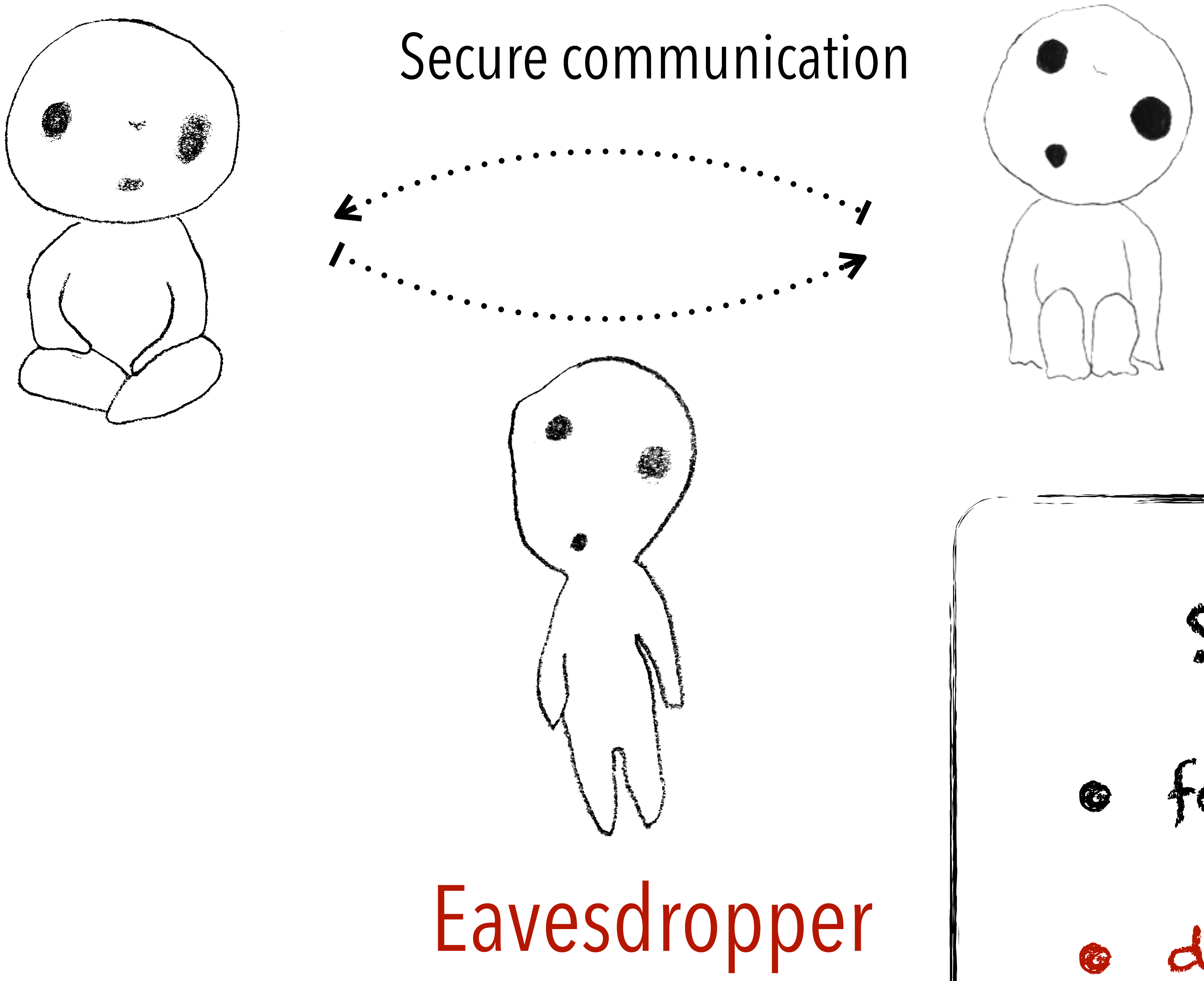
Start from...

- factorisation
- discrete logarithm
- lattice problems
- isogeny problems
- ...

... to obtain:

- encryption schemes
- signature schemes
- hash functions
- ...

Hard problems for Cryptography



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Start from...

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- lattice problems
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... to obtain:

- encryption schemes
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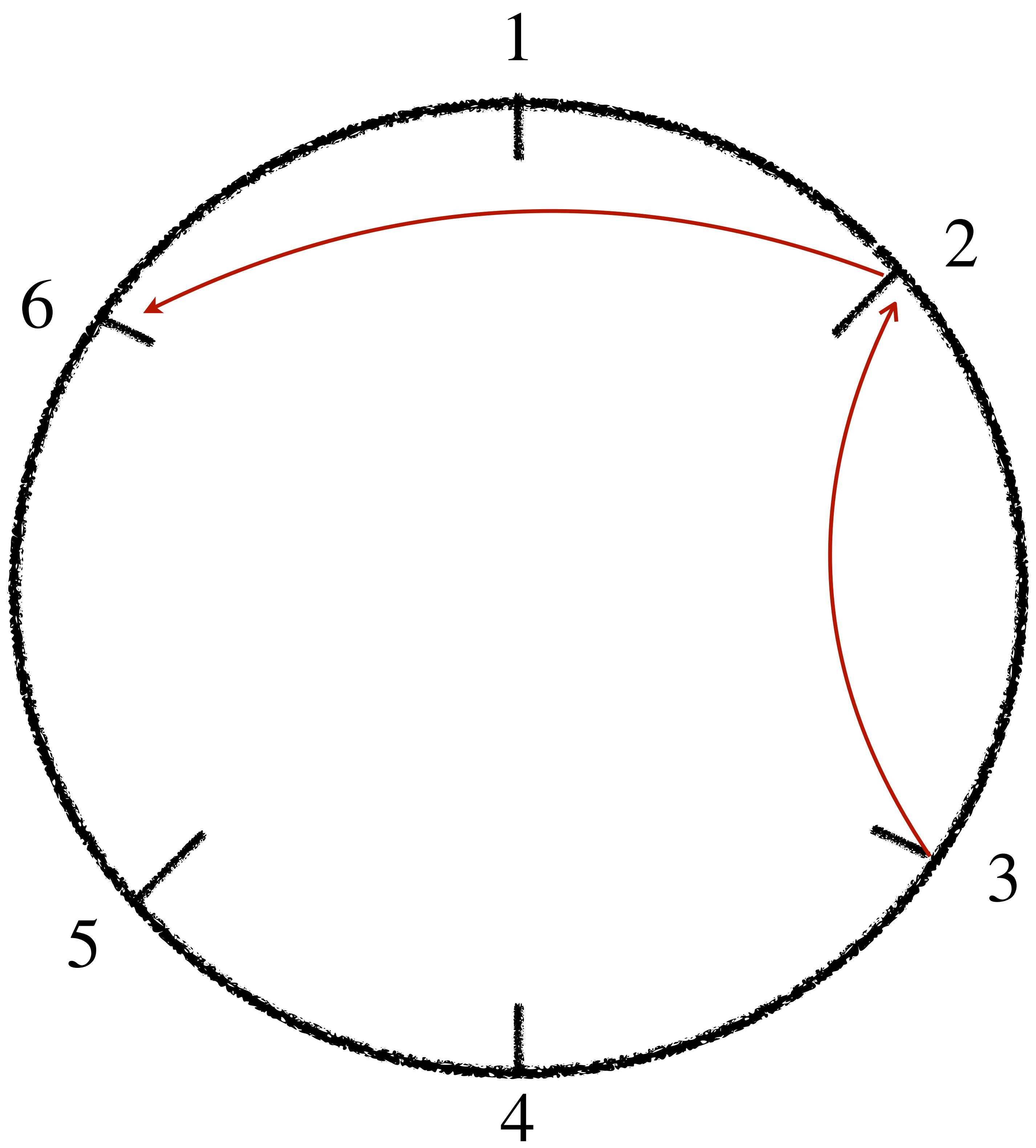
- ...

What is a discrete logarithm?

Definition: Given a finite cyclic group G of order n, a generator $g \in G$ and some element $h \in G$, the discrete logarithm of h in base g is the element $x \in [0,n)$ such that $g^x = h$.



Example: $G = \mathbb{Z}_7^\times,\ g = 3,$
$h = 6 \in \mathbb{Z}_7^\times,$

$$g^1 \equiv 3 \pmod 7$$
$$g^2 = 9 \equiv 2 \pmod 7$$
$$g^3 = 27 \equiv 6 \pmod 7$$

The discrete logarithm of h in base g is 3.

The discrete logarithm problem (DLP)

Definition: Given a finite cyclic group G of order n, a generator $g \in G$ and some element $h \in G$, **find** the element $x \in [0,n)$ such that $g^x = h$.

Computing the inverse, a modular exponentiation is easy:

$$g^x = \underbrace{g \cdot g \cdot \cdots \cdot g}_{x}$$

algorithms in $O(\log(x))$

Solving DLP can be hard (depending on the group G):

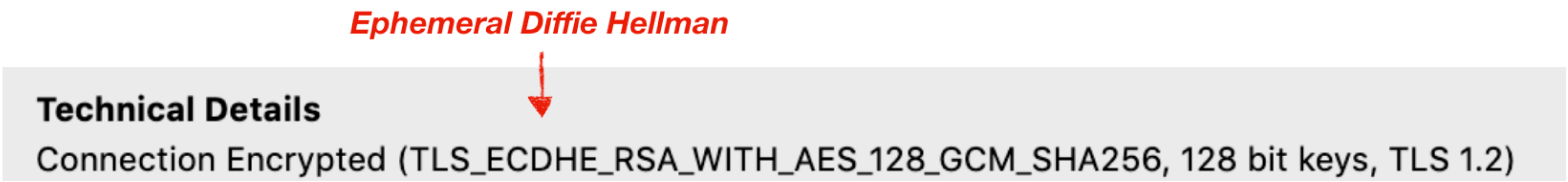
$$h = \underbrace{g \cdot g \cdot \cdots \cdot g}_{??}$$

Why do we care about discrete logarithms?

Many protocols use modular exponentiation where the exponent is a secret.

Example 1: Diffie-Hellman key exchange [DH76]

- Public data: $g, g^a, g^b \in G$
- Shared key: $g^{ab} \in G$



Example 2: pairing-based protocols

- Identity-based encryption/signature schemes [BF01], [CC03]
- Short signature schemes (eg, BLS signatures [BLS01])

Security based on assumptions that become false if DLP is broken.

[DH76]: W. Diffie, M. Hellman, New directions in cryptography. Trans. Info. Theory, 1976

[BF01]: D. Boneh, M. Franklin, Identity-based encryption from Weil pairing. Crypto'01

[CC03]: J. Cha, J. Cheon, An identity-based signature from gap Diffie-Hellman groups. PKC'03

[BLS01]: D. Boneh, B. Lynn, H. Shacham, Short signatures from the Weil pairing. Asiacrypt'01

In my work

How can we assess the security of protocols in which a modular exponentiation involving a secret exponent is performed?

- Estimate the hardness of DLP in the groups considered by the protocols.
- Look at implementation vulnerabilities during fast exponentiation.

An example: EPID protocol in Intel SGX

- What is EPID? a protocol to allow remote attestation of a hardware platform without compromising the device's identity.

- The protocol includes a signing algorithm that uses pairings.

  - secret key includes the element $f \in_R \mathbb{Z}_q$

- How can we recover f?

  - During the protocol, consider a random secret nonce $r \in \mathbb{Z}_q$

  - Compute an exponentiation X^r

  - Outputs the element $s \leftarrow r + cf$ ($c =$ hash of known values)

How can we recover the secret f ?

Since $s \leftarrow r + cf$, if we recover r , we directly get f.

The protocol uses a 256-bit elliptic curve Fp256BN (embedding degree 12).

If we have as target X^r :

1. Solve DLP to find exponent r in 3072-bit finite field $\mathbb{F}_{p^{12}}$.

2. Look at implementation vulnerabilities during the computation of X^r.

In my work

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- Estimate the hardness of DLP in the groups considered by the protocols.
- ~~Look at implementation vulnerabilities during fast exponentiation.~~

The discrete logarithm problem over finite fields

Definition: Given a finite cyclic group G of order n, a generator $g \in G$ and some element $h \in G$, find the element $x \in [0,n)$ such that $g^x = h$.

What group G should be considered?

$$(\cancel{\mathbb{Z}/n\mathbb{Z}}, +)$$

- Prime finite fields $\mathbb{F}_p^\times$
- Finite fields $\mathbb{F}_{p^n}^\times$
- Elliptic curves over finite fields $\mathcal{E}(\mathbb{F}_p)$
- Genus 2 hyperelliptic curves

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Evaluating the hardness of DLP over $\mathbb{F}_{p^n}$

- Many different algorithms to solve DLP in $\mathbb{F}_{p^n}$.
- Their complexities depend on the relation between the characteristic p and the extension degree n.

A useful notation: the L-notation $$\boxed{L_{p^n}(\alpha, c) = \exp((c + o(1))\log(p^n)^\alpha \log\log(p^n)^{1-\alpha})}$$

for $0 \leqslant \alpha \leqslant 1$ and $c > 0$.

For complexities:

- When $\alpha \to 0$: $\exp(c \log\log p^n) \approx (\log p^n)^c$, polynomial-time
- When $\alpha \to 1$: p^{cn}, exponential-time

In the <u>middle</u>: subexponential-time

Three families of finite fields

Finite field $\mathbb{F}_{p^n}$ with $p = L_{p^n}(\alpha, c)$

- Different algorithms are used in the different areas.
- Algorithms don't have the same complexity in each area.

Index calculus algorithms

Consider a finite field $\mathbb{F}_{p^n}$

Factor basis: $\mathscr{F} =$ small set of small elements

Three main steps:

- Relation collection: find relations between the elements of $\mathscr{F}$.
- Linear algebra: solve a system of linear equations where the unknowns are the discrete logarithms of the elements of $\mathscr{F}$.
- Individual logarithm/Descent: for a target element $h \in \mathbb{F}_{p^n}^{\times}$, compute the discrete logarithm of h.

A lot of algorithms

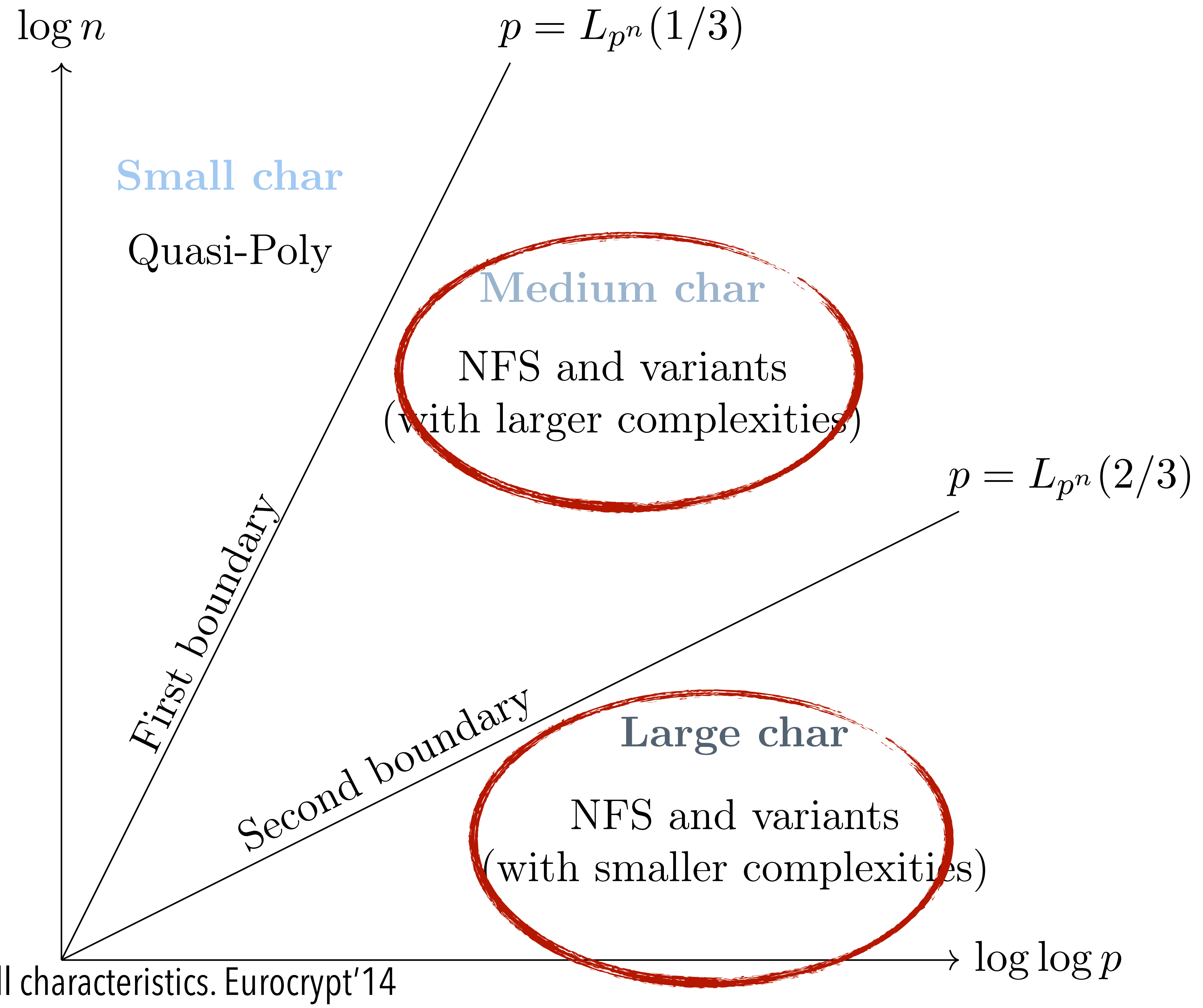
- Small characteristics: Quasi-Polynomial algorithms [BGJT14, KW19] (with only a descent step) and Function Field Sieve [Adl94]

- Medium and large characteristics: Number Field Sieve (NFS) [Gor93] and its variants

We focus on medium and large characteristic finite fields.

Why?

Finite fields used in practice for example $\mathbb{F}_{p^6}$ for MNT-6 elliptic curves in zk-SNARKS.



[Adl94]: L. Adleman, The Function Field Sieve. ANTS'94
[Gor98]: D. Gordon, Discrete Logarithms in GF(P) Using the Number Field Sieve. Journal on Discrete Mathematics'93
[BGJT14]: R. Barbulescu, P. Gaudry, A. Joux, E. Thomé, A heuristic quasi-polynomial time algorithm for discrete logarithm in finite fields of small characteristics. Eurocrypt'14
[KW19]: T. Kleinjung, B. Wesolowski, Discrete logarithms in quasi-polynomial time in finite fields of fixed characteristic. 2019

Why do we do record computations?

It is important to choose the right key size.

- Too large: needlessly expensive computations
- Too small: insecure

Agency	Date	Size of group	Size of key
NIST	2019-2030	2048	224
	> 2030	3072	256
ANSSI	2021-2030	2048	200
	> 2030	3072	200

Running-time of discrete logarithm algorithms is hard to predict.

Record computations provide information for assessing key lifetime.

A first record computation with exTNFS

- Why did we choose exTNFS?

$$n = \eta\kappa$$

$$\mathbb{F}_{p^n} = \mathbb{F}_{p^{\eta\kappa}} = \mathbb{F}_{P^\kappa}$$

Specificity	Algorithm	Medium characteristic	2nd boundary	Large characteristic
None	NFS	96	48	64
	MNFS	89.45	45.00	61.93
	TNFS	–	–	64
	MTNFS	–	–	61.93
Composite n	exTNFS	48	–	–
	MexTNFS	45.00	–	–
Special p	SNFS	$64\left(\frac{\lambda+1}{\lambda}\right)$	$\star$	32
	STNFS	–	–	32
Composite n and special p	SexTNFS	32	$\star$	32

- Main difficulty: relation collection in dimension > 2.

[KB16]: T. Kim, R. Barbulescu, Extended tower number field sieve. Crypto'16

Collecting relations in TNFS

- Relation collection: find relations between the elements of $\mathscr{F}$.

More precisely, what does this mean? What is a relation? Who is $\mathscr{F}$?

For TNFS: $R = \mathbb{Z}[\iota]/h(\iota)$

In our computation:

- $n = 6 = 3 \times 2$
- $\deg h = \eta = 3$
- $h = \iota^3 - \iota + 1$

$$\begin{tikzcd}
& R[X] \arrow[dl] \arrow[dr] & \\
K_1 \supset R[X]/(X^4+1) \arrow[dr] & & K_2 \supset R[X]/(aX^2+bX+c) \arrow[dl] \\
& (R/pR)[X]/(aX^2+bX+c) \cong \mathbb{F}_{p^6} &
\end{tikzcd}$$

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$$\begin{tikzcd}
& R[X] \arrow[dl] \arrow[dr] \arrow[dd, bend right] \arrow[dd, bend left] & \\
K_1 \supset R[X]/(X^4+1) \arrow[dr] & & K_2 \supset R[X]/(aX^2+bX+c) \arrow[dl] \\
& \mathbb{F}_{p^6} &
\end{tikzcd}$$

Collecting relations in TNFS: what is a relation?

$$R = \mathbb{Z}[\imath]/(\imath^3 - \imath + 1)$$

$$\begin{tikzcd}
 & \phi(\imath, X) = a(\imath) - b(\imath)X \in R[X] \arrow[dl] \arrow[dr] & \\
K_1 \supset R[X]/(X^4+1) \arrow[dr] & & K_2 \supset R[X]/(aX^2+bX+c) \arrow[dl] \\
 & \mathbb{F}_{p^6} &
\end{tikzcd}$$

$$\phi(\imath, \alpha_1) = a(\imath) - b(\imath)\alpha_1$$

Test $N(\phi(\imath, \alpha_1))$ for B-smoothness:

$\longrightarrow$ prime factors smaller than B

Equality in finite field = Relation

Collecting relations in TNFS: what is a relation?

- Relation collection: find relations between the elements of $\mathscr{F}$.

$$\begin{tikzcd} & R[X] \arrow[dl] \arrow[dr] & \\ K_1 \supset R[X]/(X^4+1) \arrow[dr] & & K_2 \supset R[X]/(aX^2+bX+c) \arrow[dl] \\ & \mathbb{F}_{p^6} & \end{tikzcd}$$

Who is $\mathscr{F}$?

Prime ideals of small norm in the ring of integers of the intermediate number fields

$$\prod \mathfrak{p}_i^{e_i} \text{ " = " } \prod \mathfrak{q}_j^{f_j}$$

Collecting relations in TNFS

Relation collection looks for a set of linear polynomials $\phi(\iota, X) = a(\iota) - b(\iota)X \in R[X]$

1. with bounded coefficients → $c \in \mathcal{S}$ where $\mathcal{S}$ is known as the sieving region.

2. such that $N_i(a(\iota) - b(\iota)\alpha_i)$ is B-smooth → Norms divisible only by primes smaller than B:
$c \in$ intersection of suitably constructed lattices $\mathcal{L}$

Concretely, let:

$$a(\iota) = a_0 + a_1\iota + a_2\iota^2$$

$$b(\iota) = b_0 + b_1\iota + b_2\iota^2$$

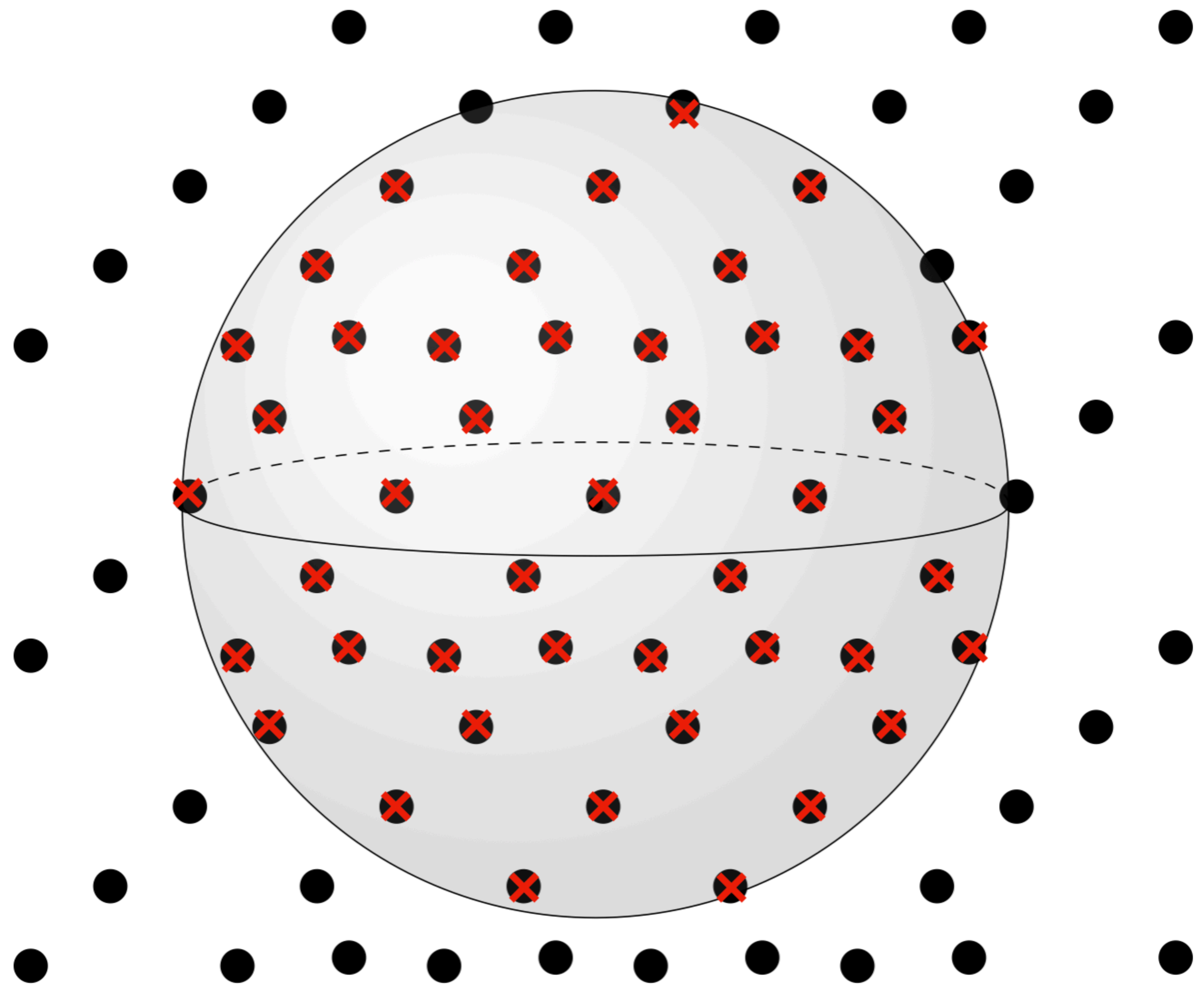
Goal: find vectors $c = (a_0, a_1, a_2, b_0, b_1, b_2) \in \mathbb{Z}^6$ such that

A new sieving region

Goal: find $c \in \mathcal{S} \cap \mathcal{L}$

What is the dimension of $\mathcal{S}$? $d = 2\eta = 6$

We look at TNFS so dimension > 2 (since $\eta \geq 2$) and $\mathcal{S} = 6$-**sphere** (**ℓ_2-norm).**



Enumerating in $\mathscr{S} \cap \mathscr{L}$

- Concretely what is $\mathscr{L}$?

A lattice that describes the divisibility of the ideals by an ideal $\mathfrak{Q}$, known as a special-q ideal and a prime ideal $\mathfrak{p}$ in the intermediate number fields.

for many $\mathfrak{p}'s$

- The outputs of the enumeration are thus …

…vectors corresponding to (a, b) pairs whose norms are divisible by $N(\mathfrak{Q})$ and $N(\mathfrak{p})$.

Why? high probability of B-smoothness

Schnorr-Euchner's enumeration [SE94]

- Input: a lattice basis $\mathbf{b}_1, \cdots, \mathbf{b}_d$
- Output: shortest non-zero lattice vector

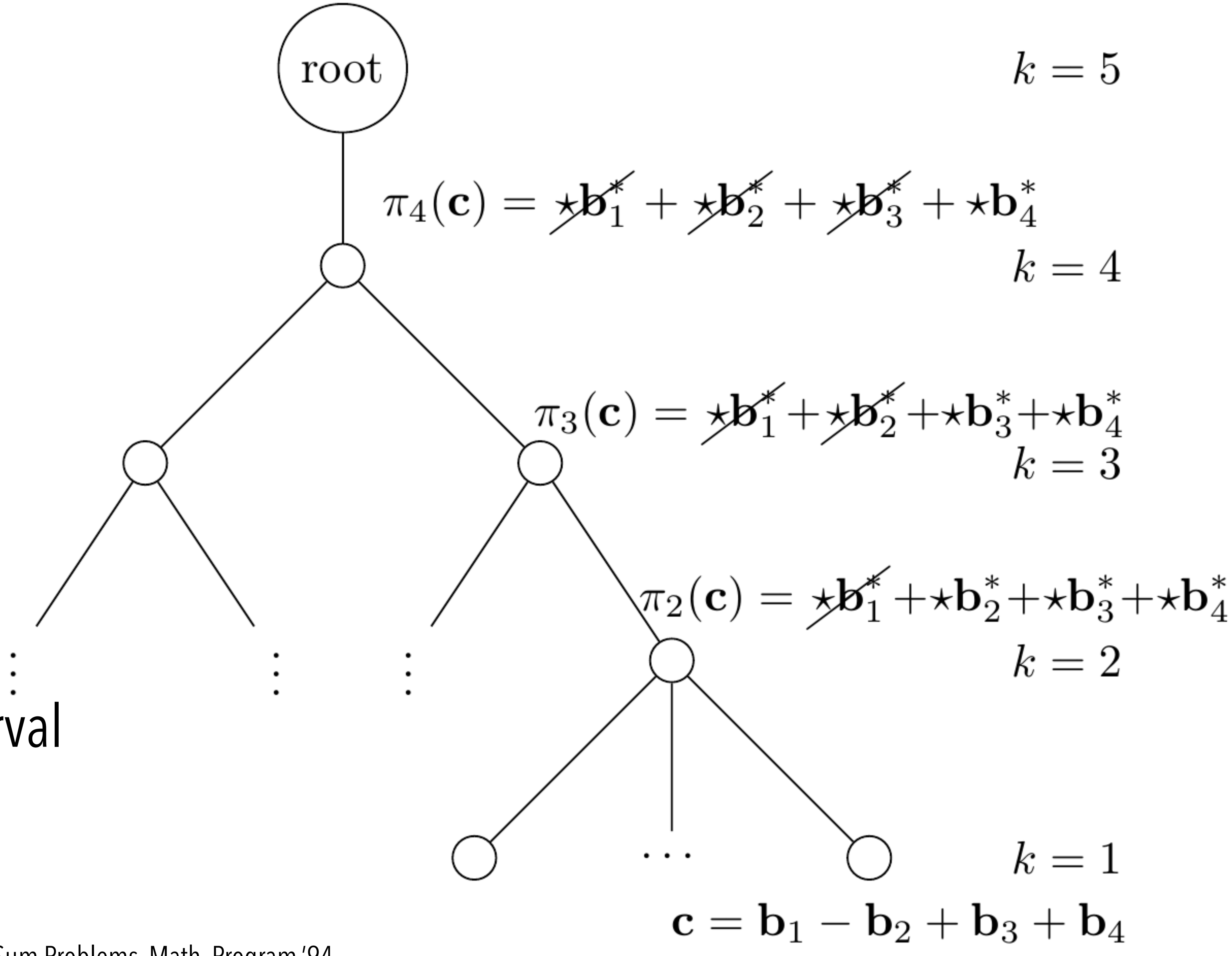
Idea:

1. Construct an enumeration tree

2. Consider projections of the lattice

3. At each level of the tree, enumerate in an interval

4. Depth-first search in the tree



[SE94]: C-P. Schnorr, M. Euchner, Lattice Basis Reduction: Improved Practical Algorithms and Solving Subset Sum Problems. Math. Program.'94

Schnorr-Euchner's enumeration [SE94]

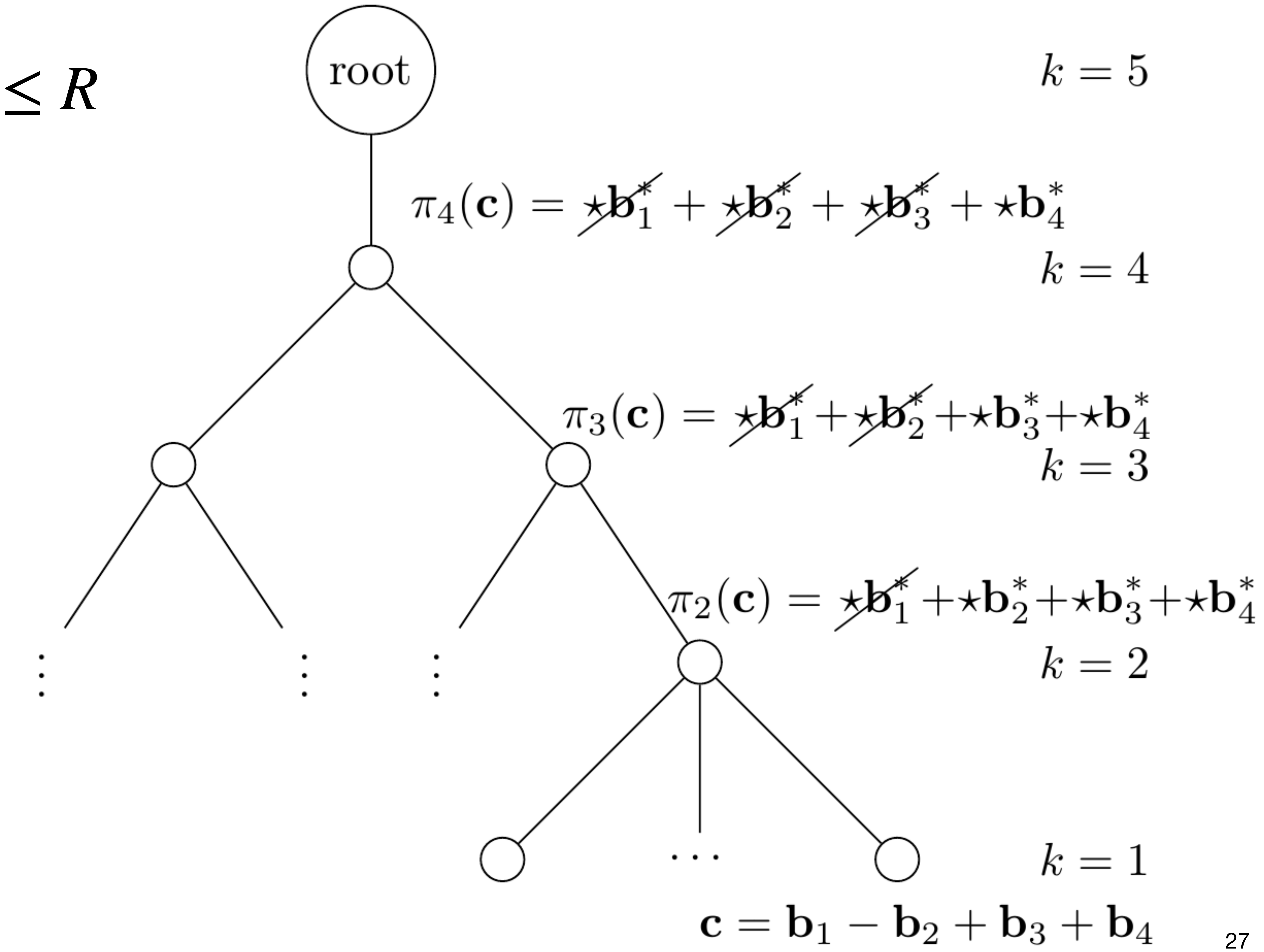
- Input: a lattice basis $\mathbf{b}_1, \cdots, \mathbf{b}_6$
- Output: vectors $c = \sum v_i \mathbf{b}_i$ such that $||c|| \leq R$

Idea:

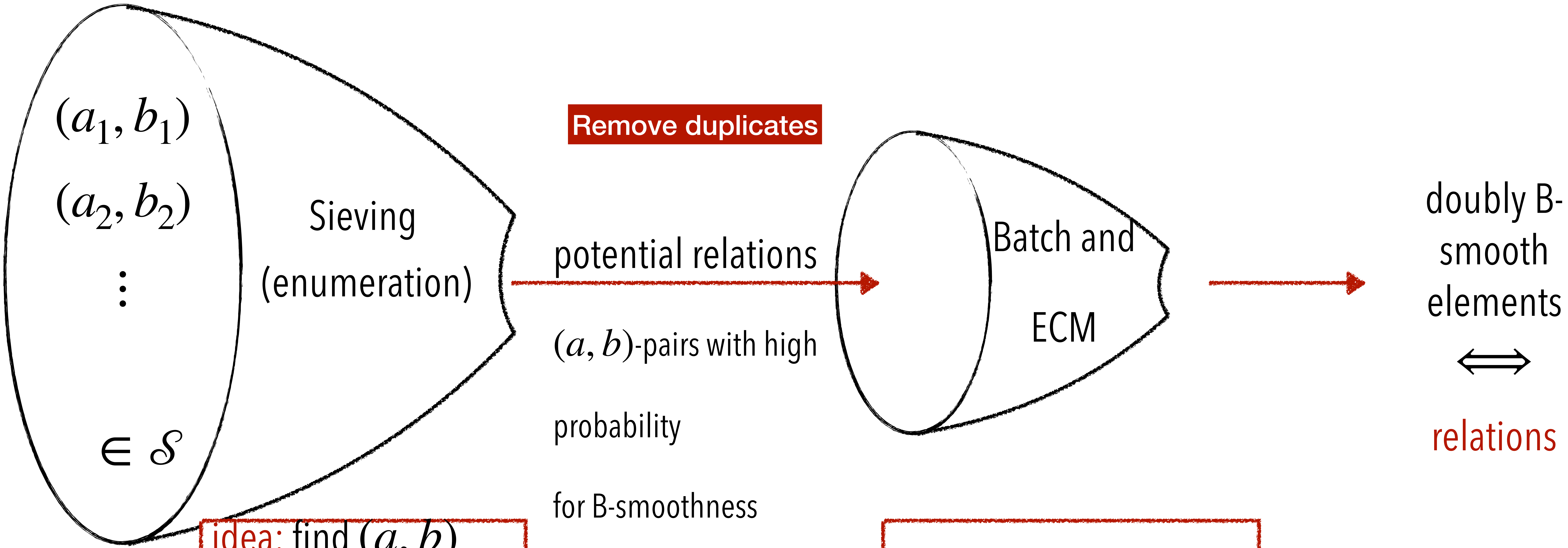
1. Construct an enumeration tree

2. Consider projections of the lattice

3. Exhaustive search of the coefficients v_i



Relation collection all together



idea: find (a, b) -pairs in $\mathcal{S}$ intersect many lattices $\mathcal{L}_{\mathfrak{p}}$.

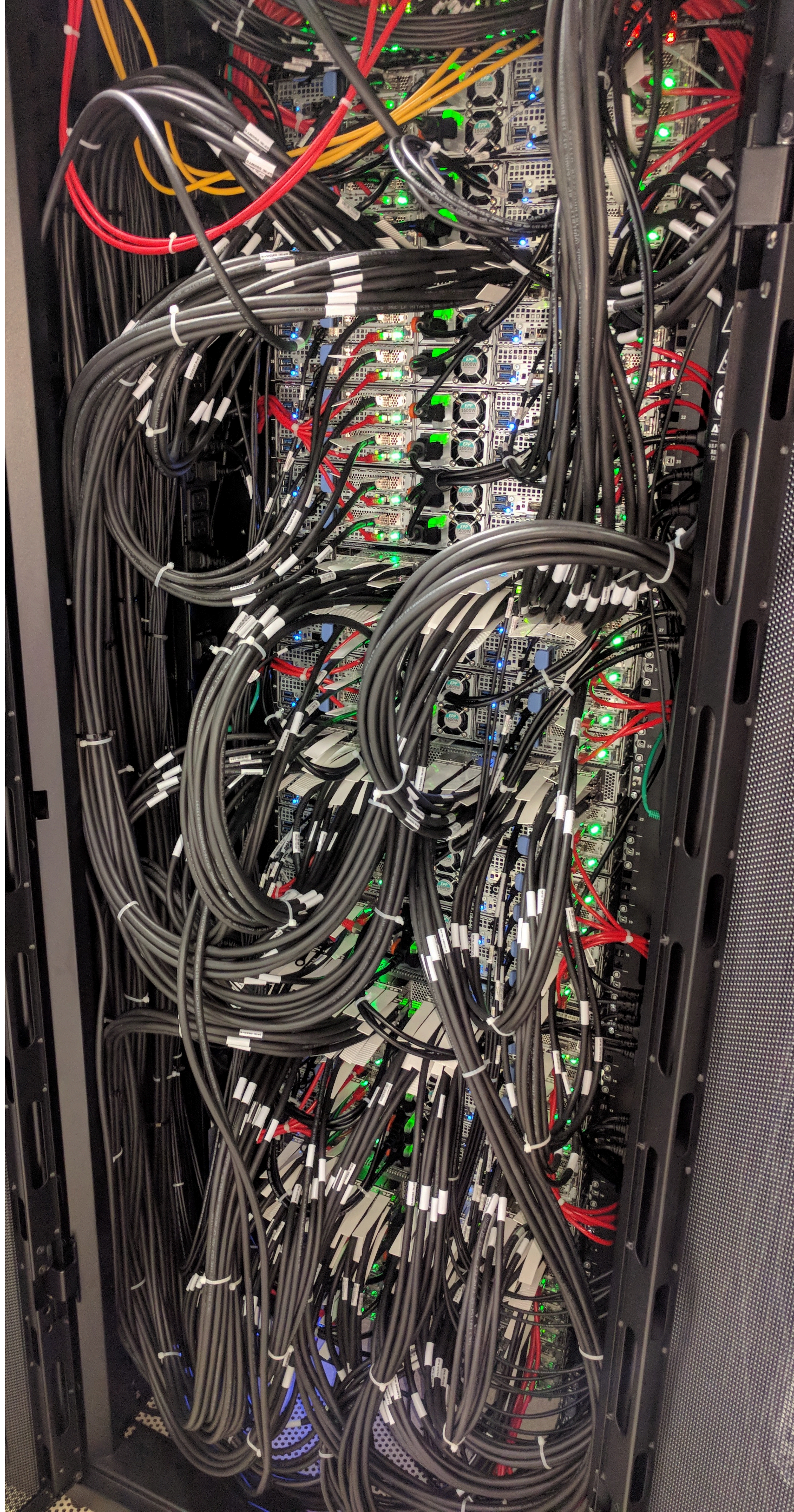
idea: test B-smoothness of the potential relations

What we needed for a record computation

- A fast sieving algorithm in dimension > 2.
- Identifying and removing duplicate relations.
- Adapting Schirokauer maps (virtual logarithms) to TNFS context.
- Glue-code to branch into CADO-NFS.
- A nice target: $\mathbb{F}_{p^6}$.

↑
in theory…

in practice… ⟶ grvingt



Our 521-bit record computation

$\mathbb{F}_{p_{87}^6}$

First boundary | Second boundary

0	Small char	$\frac{1}{3}$	Medium char	$\frac{2}{3}$	Large char	1	α
$\mathbb{F}_{2^{1024}}$		$\mathbb{F}_{p_5^{205}}$		$\mathbb{F}_{p_{30}^{34}}$		$\mathbb{F}_{p_{1024}}$	

Total computation time (core hours):

Relation Collection	Linear algebra	Schirokauer maps	Descent	Overall time
23,300	1,403	40	55	24,798

Focus on relation collection:

Parameters	[GGMT17]	[MR21]	This work
Algorithm	NFS	NFS	TNFS
Field size (bits)	422	423	521
Sieving dimension	3	3	6
Sieving time	201,600	69,120	**23,300**

[GGMT17]: L. Grémy, A. Guillevic, F. Morain, E. Thomé, Computing discrete logarithm in Fp6. Sac'17
[MR21]: G. McGuire, O. Robinson, Lattice Sieving in three dimensions for discrete log in medium characteristic. Journal of mathematical cryptology'21

A discrete logarithm

Finite field: $\mathbb{F}_{p^6}$ with 87-bit prime p, generator $g = x + \iota$

$$\text{target} = (31415926535897932384626433 + 83279502884197169399375105\iota + 82097494459230781640628620\iota^2) + x(89986280348253421170679821 + 48086513282306647093844609\iota + 55058223172535940812848111\iota^2)$$

$$\log(\text{target}) = 76272808168753222977667479701383785303538529763154 98$$

Thank you for your attention!

A discrete logarithm (in more details)

$p =$ 0x6fb96ccdf61c1ea3582e57 (87-bit prime) $\quad n = 6$

$$\mathbb{F}_{p^6} = \mathbb{F}_{p^3}[x]/(x^2 + 6441772330699146441962235 3x + 1)$$

Irreducible factor mod p, here f2

target = $a(\imath) + xb(\imath) \in \mathbb{F}_{p^6}$ with: $a(\imath), b(\imath)$ of degree 2 and coefficients $< p$.

$$\text{target} = (31415926535897932384626433 + 83279502884197169399375105\imath$$
$$+82097494459230781640628620\imath^2) + x(89986280348253421170679821$$
$$+48086513282306647093844609\imath + 55058223172535940812848111\imath^2)$$

generator = $x + \imath$

log(target) = 7627280816875322297766747970138378530353852976315498

Verification: $(x + \imath)^{\log(\text{target})} = \text{target} \pmod{\ell\text{-th powers}}$

Choice of subgroup

Initial target: $\mathbb{F}_{p^6}$ $\xrightarrow{\text{Pohlig-Hellman:}}$ Prime order subgroup of order $\ell \mid p^6 - 1$

We have the following factorisation: $p^6 - 1 = (p-1)(p+1)(p^2+p+1)(p^2-p+1)$

- $p - 1 = |\mathbb{F}_p^\times|$ If g and h are of order $\ell \mid p-1 \Rightarrow g, h \in \mathbb{F}_p^\times \Rightarrow$ NFS in $\mathbb{F}_p$ of 87 bits
- $p + 1 = |\mathbb{F}_{p^2}^\times| / |\mathbb{F}_p^\times|$ If g and h are of order $\ell \mid p+1 \Rightarrow g, h \in \mathbb{F}_{p^2}^\times \Rightarrow$ NFS in $\mathbb{F}_{p^2}$ of 175 bits
- $p^2 + p + 1 = |\mathbb{F}_{p^3}^\times| / |\mathbb{F}_p^\times|$ If g and h are of order $\ell \mid p^2+p+1 \Rightarrow g, h \in \mathbb{F}_{p^3}^\times \Rightarrow$ NFS in $\mathbb{F}_{p^3}$ of 261 bits
- $p^2 - p + 1$: 6th-cyclotomic subgroup Here, we can't go in a smaller subgroup…

Attention: it is not the largest subgroup!

Multiplicative group of a finite field

- The non-zero elements of a finite field form a multiplicative group.

- This group is cyclic, so all non-zero elements can be expressed as powers of a single element called a primitive element of the field.

Example 1: prime order finite fields: $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$

multiplicative group: $\mathbb{F}_p^\times = \{1, 2, \cdots, p-1\} = \mathbb{F}_p \backslash \{0\}$

Example 2: non-prime order finite fields: $\mathbb{F}_{p^n} \cong \mathbb{F}_p[X]/(P)$

--> elements are polynomials over $\mathbb{F}_p$ whose degree is less than n.

multiplicative group: $\mathbb{F}_{p^n}^\times = \{\text{invertible polynomials}\} = \mathbb{F}_{p^n} \backslash \{0\}$

Number field vs Function fields

Number field:

Finite extension of $\mathbb{Q}$

$\mathbb{Q} = \{p/q : p, q \text{ integers}\}$

$K = \mathbb{Q}[x]/(f)$

Example: $f = x^2 - d$

$K = \{x + y\sqrt{d} : x, y \in \mathbb{Q}\}$

Factor basis: prime ideals in $\mathcal{O}_K$

B-smoothness: compute norm of ideal = integer (from a resultant)

Function field:

Finite extension of $\mathbb{F}_p(\imath)$

$\mathbb{F}_p(\imath) = \{p(\imath)/q(\imath) : p(\imath), q(\imath) \in \mathbb{F}_p[\imath]\}$

$K = \mathbb{F}_p(\imath)[x]/(f)$

Example: $f = x^2 - (\imath^3 + 2\imath - 3)$

$K = \{x_0 + x_1\sqrt{\imath^3 + 2\imath - 3} : x_0, x_1 \in \mathbb{F}_p(\imath)\}$

Factor basis: prime ideals in $\mathcal{O}_K$

B-smoothness: compute norm of ideal = univariate polynomial (from a bivariate resultant)

Why do we choose a d-sphere?

Assumption: *size of norms depends only on size of vector coordinates.*

The norm for $c' \in C \backslash S_d(R)$ is greater than the norm for $c \in S_d(R)$.

When $d \to \infty$:

Difference in norms increases!

Conclusion: choosing $S_d(R)$ leads to smaller norms.